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Brief Communication

Bicovariant DeRham Cohomology of $SU_q(2)$

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Abstract

Bicovariant differential forms over the quantum group $SU_q(2)$ are considered. Their multiplet structure is determined under the action of $SU_q(2)$. This gives a basis of the forms. The exterior derivative of forms and elements of the Hopf algebra $\mathcal{A}$, which is generated by the matrix elements $M^i{}_j$, is explicitly calculated. Then the closed and exact forms are compared. This gives the bicovariant DeRham Cohomology of $SU_q(2)$.

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1. Introduction

In this work I consider especially the quantum group $SU_q(2)$. It has already been investigated in great detail in [Weich,Wor1]. In [Wor1] the DeRham Cohomology of $SU_q(2)$ has been determined. But this was based on a differential calculus of $SU_q(2)$, which is not a bicovariant bimodule over the Hopf algebra $\mathcal{A}(SU_q(2))$. In this work now the DeRham Cohomology is determined with the help of the bicovariant differential calculus from [CSWW].

2. The quantum group $SU_q(2)$

The notion of a quantum group has been investigated in [FRT] and [Drin]. My notation is guided by [CSWW], specialized to the case $SU_q(2)$.

The main structure of a quantum group is the structure of a Hopf algebra. The quantum group $SU_q(2)$ is the Hopf algebra $\mathcal{A} = \mathcal{A}(SU_q(2))$ generated by the matrix

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elements $M^i{}_j$, $i,j = 1,2$, and with coproduct Δ, counit ε and antipode κ and with the following commutation relations:

$$\widehat{R}_q{}^{ij}{}_{j'i'} M^{j'}{}_{j''} M^{i'}{}_{i''} = M^i{}_{i'} M^j{}_{j'} \widehat{R}_q{}^{i'j'}{}_{j''i''} , \tag{1}$$

where the R-matrix is defined as

$$\widehat{R}_q = (\widehat{R}_q{}^{ij}{}_{kl}) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} .$$

A convenient basis over the field of complex numbers for the algebra $\mathcal{A}$ is [Wor1]

$$\left\{ (M^1{}_1)^k (M^1{}_2)^l (M^2{}_1)^m, (M^1{}_2)^l (M^2{}_1)^m (M^2{}_2)^n : k,l,m \in \mathbb{N},\ n \in \mathbb{N}\backslash\{0\} \right\} . \tag{2}$$

With the help of (1) and $\det M = M^1{}_1 M^2{}_2 - q M^1{}_2 M^2{}_1 = 1$ it is possible to write every monom in the matrix elements in this basis.

$SU_q(2)$ has also a corresponding Exterior Algebra $\Gamma^\wedge_{\mathrm{Ad}}$ with left- and right-coaction Δ_L and Δ_R. $\rho \in \Gamma^\wedge_{\mathrm{Ad}}$ is called *left-invariant* if $\Delta_L(\rho) = 1 \otimes \rho$ and *right-invariant* if $\Delta_R(\rho) = \rho \otimes 1$. The invariant elements form linear subspaces ${}_{\mathrm{inv}}\Gamma^\wedge_{\mathrm{Ad}}$ and $\Gamma^\wedge_{\mathrm{Ad\,inv}}$ of $\Gamma^\wedge_{\mathrm{Ad}}$, respectively. If $\{\eta^J \in \Gamma^\wedge_{\mathrm{Ad\,inv}} : J \text{ of an index set}\}$ is a basis of the right-invariant elements then every element $\rho \in \Gamma^\wedge_{\mathrm{Ad}}$ can be written as

$$\rho = \sum_J a_J \eta^J , \quad a_J \in \mathcal{A} . \tag{3}$$

The bicovariant bimodule Γ_{Ad} of $SU_q(2)$ is constructed in a way such that the right-invariant basis transforms as the adjoint representation [CSWW]. This means that the right-invariant basis of $\Gamma_{\mathrm{Ad\,inv}}$, denoted as $\Theta^i{}_j$, has to transform under left-coaction as

$$\Delta_L(\Theta^i{}_j) = M^i{}_{i'} \kappa(M^{j'}{}_j) \otimes \Theta^{i'}{}_{j'} .$$

If one uses a different right-invariant basis

$$\Theta^{ij} := \Theta^i{}_{j'} \varepsilon^{j'j} ,$$

then it will transform as

$$\Delta_L(\Theta^{ij}) = M^i{}_k M^j{}_l \otimes \Theta^{kl} .$$

The right-invariant basis $\{\Theta^{11}, \Theta^{12}, \Theta^{21}, \Theta^{22}\}$ can be split into irreducible parts, i.e. in a singlet

$$\Theta^{ij}_X := \mathcal{P}_X{}^{ij}{}_{kl}\, \Theta^{kl} = -\frac{1}{[\![2]\!]} \varepsilon^{ij} X , \quad X := \varepsilon_{kl} \Theta^{kl} , \quad \Delta_L(X) = 1 \otimes X ,$$

and a triplet, that transforms like the adjoint representation:

$$\Theta^{ij}_{\mathrm{Ad}} := \mathcal{P}_{\mathrm{Ad}}{}^{ij}{}_{kl}\, \Theta^{kl} , \qquad \Delta_L(\Theta^{ij}_{\mathrm{Ad}}) = M^i{}_k M^j{}_l \otimes \Theta^{kl}_{\mathrm{Ad}} .$$

This triplet consists of e.g. $\Theta^{11}_{\mathrm{Ad}}$, $\Theta^{12}_{\mathrm{Ad}}$ and $\Theta^{22}_{\mathrm{Ad}}$, because the fourth element $\Theta^{21}_{\mathrm{Ad}}$ is linearly dependent: $\Theta^{21}_{\mathrm{Ad}} = q^{-1} \Theta^{12}_{\mathrm{Ad}}$.

For the DeRham Cohomology one needs a basis of the differential forms over the field of complex numbers. For this it is useful to know the irreducible multiplet structure. The results are:

- $\{\Theta^{11}_{\mathrm{Ad}}, \Theta^{12}_{\mathrm{Ad}}, \Theta^{22}_{\mathrm{Ad}}, \widetilde{X}\}$ is a basis of $\Gamma_{\mathrm{Ad\,inv}}$,
- $\{\Lambda^{11}, \Lambda^{12}, \Lambda^{22}, \chi\Lambda^{11}, \chi\Lambda^{12}, \chi\Lambda^{22}\}$ is a basis of $\Gamma^{\wedge 2}_{\mathrm{Ad\,inv}}$,
- $\{\Phi^{11}_5, \Phi^{12}_5, \Phi^{22}_5, \Phi_2\}$ is a basis of $\Gamma^{\wedge 3}_{\mathrm{Ad\,inv}}$,
- Ω_2 is a basis of $\Gamma^{\wedge 4}_{\mathrm{Ad\,inv}}$,
- $\Gamma^{\wedge 5}_{\mathrm{Ad}} = 0$. (4)

For instance with $\widetilde{X} := \sqrt{q}\,X$,

$$\begin{aligned}
\Lambda^{11} &:= \Theta^{11}_{\mathrm{Ad}} \otimes \Theta^{12}_{\mathrm{Ad}} - q^2 \Theta^{12}_{\mathrm{Ad}} \otimes \Theta^{11}_{\mathrm{Ad}}, & \chi\Lambda^{11} &:= \Theta^{11}_{\mathrm{Ad}} \otimes \widetilde{X},\\
\Lambda^{12} &:= q(\Theta^{11}_{\mathrm{Ad}} \otimes \Theta^{22}_{\mathrm{Ad}} - \Theta^{22}_{\mathrm{Ad}} \otimes \Theta^{11}_{\mathrm{Ad}}) & \chi\Lambda^{12} &:= \Theta^{12}_{\mathrm{Ad}} \otimes \widetilde{X},\\
&\quad - (q^2 - q^{-2})\Theta^{12}_{\mathrm{Ad}} \otimes \Theta^{12}_{\mathrm{Ad}}, & &\\
\Lambda^{22} &:= \Theta^{12}_{\mathrm{Ad}} \otimes \Theta^{22}_{\mathrm{Ad}} - q^2 \Theta^{22}_{\mathrm{Ad}} \otimes \Theta^{12}_{\mathrm{Ad}}, & \chi\Lambda^{22} &:= \Theta^{22}_{\mathrm{Ad}} \otimes \widetilde{X}.
\end{aligned}$$

The details are given in [G].

3. The exterior derivative $\mathrm{d} : \mathcal{A} \longrightarrow \Gamma_{\mathrm{Ad}}$

The exterior derivative $\mathrm{d} : \mathcal{A} \longrightarrow \Gamma_{\mathrm{Ad}}$ is defined as

$$\mathrm{d}a := \frac{1}{\mathcal{N}}(Xa - aX) = \frac{1}{\mathcal{N}\sqrt{q}}(\widetilde{X}a - a\widetilde{X}), \quad a \in \mathcal{A},$$

Let $a \in \mathcal{A}$. In the $\mathbb{C}$-Basis (2) of $\mathcal{A}$, a is written as

$$a = \sum_{l,m=0}^{\infty} \left[\sum_{k=0}^{\infty} \alpha_{klm0} (M^1{}_1)^k (M^1{}_2)^l (M^2{}_1)^m + \sum_{n=1}^{\infty} \alpha_{0lmn} (M^1{}_2)^l (M^2{}_1)^m (M^2{}_2)^n \right]$$

with coefficients $\alpha_{klm0}, \alpha_{0lmn} \in \mathbb{C}$, where only a finite number are nonzero. In the right-invariant basis $\{\Theta^{11}_{\mathrm{Ad}}, \Theta^{12}_{\mathrm{Ad}}, \Theta^{22}_{\mathrm{Ad}}, \widetilde{X}\}$ the derivative of a is

$$\mathrm{d}a = \frac{1}{\sqrt{q}(q - q^{-1})\mathcal{N}_0} \times \left[(a * f_{41})\Theta^{11}_{\mathrm{Ad}} + (a * f_{42})\Theta^{12}_{\mathrm{Ad}} + (a * f_{43})\Theta^{22}_{\mathrm{Ad}} + (a * f_{44} - a)\widetilde{X} \right],$$

for instance with

$$
\begin{aligned}
a * f_{41} = -(q - q^{-1}) \sum_{l,m=0}^{\infty} \Bigg[\sum_{k=0}^{\infty} \Big\{ & \alpha_{k+1\, l+1\, m\, 0}\, q^{k+l+1} [\![l+1]\!] \\
& + \alpha_{k+1\, l\, m-1\, 0}\, q^{l} [\![k+l+1]\!] \Big\} (M^1{}_1)^k (M^1{}_2)^l (M^2{}_1)^m \\
& + \sum_{n=1}^{\infty} \alpha_{0\, l+1\, m\, n-1}\, q^{-m} [\![l+1]\!] (M^1{}_2)^l (M^2{}_1)^m (M^2{}_2)^n \Bigg]
\end{aligned}
$$

(coefficients with negative indices are defined to be zero). For the derivative ${}^p\mathrm{d} : \Gamma_{\mathrm{Ad}}^{\wedge p} \to \Gamma_{\mathrm{Ad}}^{\wedge (p+1)}$ on p-forms one gets similar results.

4. The DeRham Cohomology $\mathrm{H}^p(SU_q(2))$

The DeRham Cohomology is defined as

$$
\mathrm{H}^p(SU_q(2)) = \frac{\ker({}^p\mathrm{d})}{\mathrm{im}({}^{p-1}\mathrm{d})}, \quad p > 0, \qquad \mathrm{H}^0(SU_q(2)) = \ker({}^0\mathrm{d}).
$$

If one writes $\ker({}^p\mathrm{d})$ and $\mathrm{im}({}^{p-1}\mathrm{d})$ in the basis elements (2) and right-invariant basis (4) one can compare both and find after some tedious calculations e.g.

$$
\ker({}^1\mathrm{d}) = \mathrm{im}({}^0\mathrm{d}) \oplus \mathbb{C}\widetilde{X} \quad \Rightarrow \quad \mathrm{H}^1(SU_q(2)) \cong \mathbb{C}.
$$

The results are:

$$
\begin{aligned}
&\mathrm{H}^0(SU_q(2)) \cong \mathbb{C}, \\
&\mathrm{H}^1(SU_q(2)) \cong \mathbb{C}, \\
&\mathrm{H}^2(SU_q(2)) = 0, \\
&\mathrm{H}^3(SU_q(2)) \cong \mathbb{C}, \\
&\mathrm{H}^4(SU_q(2)) \cong \mathbb{C}.
\end{aligned}
$$

Comparing this result with $SU(2) \times U(1)$,

$$
\mathrm{H}^p(SU(2) \times U(1)) \cong \begin{cases} \mathbb{C} & \text{for } p = 0,\ p = 1,\ p = 3 \text{ or } p = 4\ , \\ 0 & \text{otherwise}, \end{cases}
$$

one sees that they are identical. This can be understood by considering the classical limit $q \to 1$. The right-invariant basis $\{\Theta_{\mathrm{Ad}}^{11}, \Theta_{\mathrm{Ad}}^{12}, \Theta_{\mathrm{Ad}}^{22}, \widetilde{X}\}$ of $\Gamma_{\mathrm{Ad\,inv}}$ goes over into the right-invariant basis of the classical bimodule $\Gamma_{SU(2)}^{\wedge}$ over $SU(2)$ plus a decoupled X, which can be seen as basis of $\Gamma_{U(1)}^{\wedge}$.

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References

[CSWW] U. Carow-Watamura, M. Schlieker, S. Watamura and W. Weich, Bicovariant differential calculus on quantum groups $SU_q(N)$ and $SO_q(N)$, Commun. Math. Phys. 142 (1991) 605–641.

[Drin] V.G. Drinfeld, in: *Quantum Groups, Proc. Intern. Congress of Mathematicians*, Vol. 1 (1986) pp. 798–820.

[FRT] L.D. Faddeev, N.Y. Reshetikhin and L.A. Takhtajan, Quantization of Lie groups and Lie algebras, Algebra and Analysis 1 (1989) 178 [in Russian].

[G] M. Grießl, University of Munich (LMU) preprint LMU-TPW 93-18 (1993).

[Weich] W. Weich, Ph.D. Thesis, Die Quantengruppe $SU_q(2)$—kovariante Differentialrechnung und ein quantensymmetrisches quantenmechanisches Modell, University of Karlsruhe (TH), Germany (1990).

[Wor1] S.L. Woronowicz, Twisted $SU(2)$ group. An example of non-commutative differential calculus, Publ. Res. Inst. Math. Sci. Kyoto Univ. 23 (1987) 117–181.

[Wor2] S.L. Woronowicz, Differential calculus on compact matrix pseudogroups, Commun. Math. Phys. 122 (1989) 125–170.